# Lecture 16 (Planar graph)

A graph is said to be embedded on a surface $S$ when it is drawn on $S$ so that no two edges intersect, except at endpoints. A graph is said to be planar if it can be embedded on the plane.

**Examples:**

1. A tree is embeddable on a plane.
2. Any cycle $C_n$, $n \geq 3$ is planar.
3. The $K_4$ is planar.
4. The $K_{2,3}$ is planar.
5. Draw a planar embedding of $K_5 - e$, where $e$ is any edge.
6. Draw a planar embedding of $K_{3,3} - e$, where $e$ is any edge.

**Definition 1.** Consider a planar embedding of a graph $G$. The regions on the plane defined by this embedding are called faces/regions of $G$. The unbounded face/region is called the exterior face.

**Theorem 0.1.** (Euler Formula for Planar Graph) Let $G$ be a connected planar graph with $V$ number of vertices, $E$ number of edges and $F$ number of faces. Then $V - E + F = 2$.

**Proof:** We use induction on $E$. Let $E = 1$. Then $V = 2$ and $F = 1$ and hence $V - E + F = 2$.

Suppose the result holds for all the planar-connected graphs with $E \leq n - 1$.

Let $G$ be a connected planar graph with $E = n$. Then we have two cases:

Case 1: If $G$ has no cycles. Then $G$ is a tree and hence the result is true.

Case 2: Suppose $G$ contains at least one cycle. Let $e$ be an edge in a cycle and $G' = G - e$. Then $E' = E - 1$, $V' = V$ and $F' = F - 1$. Then $E' - V' + F' = 2$ implies $V - E + F = 2$. Thus the result holds.

**Corollary 1.** If $G$ is a connected planar simple graph with $E$ edges and $V$ vertices, where $V \geq 3$, then $E \leq 3V - 6$.

**Proof:** Each face has at least 3 edges and each edge is participated in two faces. So, the number of edges $E \geq 3F/2$. Now the proof follows by $V - E + F = 2$.

**Exercise:** $K_5$ is non-planar, because here $V = 5$ and $E = 10$, using the above result leads to $E = 10 \leq 3 \times 5 - 6 = 9$, a contradiction.

**Corollary 2.** If a connected planar simple graph has $E$ edges and $V$ vertices with $V \geq 3$ and no cycle of length three, then $E \leq 2V - 4$.

**Proof:** Note that $E \geq 2F$. Then the proof follows by $V - E + F = 2$.

**Exercise:** $K_{3,3}$ is non-planar, because $K_{3,3}$ has no cycle of length 3 with $V = 6, E = 9$, which using the above result leads to $E = 9 \leq 2 \times 6 - 4 = 8$, a contradiction.

**Proof:** Note that $E \geq 2F$. Then the proof follows by $V - E + F = 2$.

**Remark.** The following result generalizes the above corollaries for a connected, planar graph having cycles of any length.

**Theorem 0.2.** *Let $G$ be a connected planar graph with $E$ edges and $V$ vertices, $V \geq 3$, such that $G$ has cycles of length $\geq g$, then $E \leq \frac{g}{g-2}(V - 2)$.*

**Proof:** Suppose the embedding has $F$ faces. Then $E \geq \frac{gF}{2} \Leftrightarrow F \leq \frac{2E}{g}$. Now following Euler formula, we get $E \leq \frac{g}{g-2}(V - 2)$.

**Definition 2.** Let $G$ be a graph. Then, a subdivision of an edge $u\,v$ in $G$ is obtained by replacing the edge by two edges $u\,w$ and $w\,v$, by introducing a new vertex $w$ of degree 2. Two graphs $G_1$ and $G_2$ are said to be homeomorphic if there is a graph $G_3$ such that both graphs $G_1$ and $G_2$ can be obtained from $G_3$ a sequence of subdivisions.

**Example.** For $n \neq m$, the path graphs $P_n$ is homeomorhic to $P_m$, as $P_n$ and $P_m$ can be obtained from $P_1$ by applying $n - 2$ and $m - 2$ numbers of subdivisions respectively. Similarly $C_n$ is homeomorphic to $C_m$ for distinct $m$ and $n$. However, $C_n$ is not homeomorphic to wheel graph $W_n$, where $V(W_n) = V(C_n) \cup \{a\} = \{v_1, v_2, \ldots, v_n\} \cup \{a\}$, and $E(W_n) = E(C_n) \cup \{av_1, av_2, \ldots, av_n\}$.

**Theorem 0.3. (Kuratowski's Theorem)** A graph is non planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or $K_5$.

**Exercise:** Petersen graph is non-planar. It can be shown that the said graph is homeomorphic to $K_{3,3}/K_5$.